

# Intersection matrices and the Johnson scheme

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## Abstract

In this paper we introduce a generating function  $F_{sk}^t(v)(z)$  which is a polynomial in terms of  $z$  whose coefficients are some intersection matrices. This provides a uniform framework in which several intersection matrices can be extracted from  $F_{sk}^t(v)(z)$ . Several nice properties of  $F_{sk}^t(v)(z)$  are derived by applying the operator  $d/dz$  and studying the operator  $zd/dz$ . In the new framework some well-known identities on intersection matrices arise as natural consequences. As an application two new bases for the Johnson scheme are constructed and the eigenvalues of a family of intersection matrices which contains the adjacency matrices of the Johnson scheme are derived. Finally, we determine the rank of some intersection matrices.

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## 1 Introduction

Let  $s$ ,  $k$ , and  $v$  be integers satisfying  $0 \leq s \leq k \leq v$ . The inclusion matrix  $W_{sk}(v)$  is a  $(0, 1)$ -matrix whose rows and columns are indexed by  $s$ -subsets and  $k$ -subsets of a  $v$ -set, respectively, and  $W_{sk}(v)(S, K) = 1$  if and only if  $S \subseteq K$ . This matrix has interesting properties and arise in many combinatorial problems particularly in design theory and extremal set theory (see [2, 5, 6, 12, 13, 15]). It satisfies several nice identities among which is

$$W_{is}(v)W_{sk}(v) = \binom{k-i}{s-i}W_{ik}(v), \quad (1)$$

which holds for  $i \leq s \leq k \leq v$  (see, e.g., [12, 13, 15]). A matrix which may be thought of as a dual for the inclusion matrix is the exclusion matrix:

$$\overline{W}_{sk}(v)(S, K) = \begin{cases} 1, & \text{if } S \cap K = \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

Usually for simplicity we drop  $v$  from the notations of  $W_{sk}(v)$ ,  $\overline{W}_{sk}(v)$ , and similar matrices introduced later in this paper. The exclusion matrix is related to inclusion matrix in some ways. Firstly, the matrix  $\overline{W}_{sk}$  is obtained from  $W_{s,v-k}$  by a permutation on its rows and columns. Secondly, it can be proved that the following nice identities hold:

$$\overline{W}_{sk} = \sum_{i=0}^s (-1)^i W_{is}^\top W_{ik}, \quad (2)$$

$$W_{sk} = \sum_{i=0}^s (-1)^i W_{is}^\top \overline{W}_{ik}. \quad (3)$$

Thirdly, these two matrices may be considered as intersection matrices in the sense that the entry  $(S, K)$  only depends on  $|S \cap K|$ . In this viewpoint, a natural generalization of both matrices is the matrix  $U$  defined as

$$U_{sk}^\ell(S, K) = \begin{cases} 1, & \text{if } |S \cap K| = \ell; \\ 0, & \text{otherwise.} \end{cases}$$

Notice that in the definition of  $U_{sk}^\ell$  the condition  $s \leq k$  is no more required. In the case  $s = k$  the above matrix is well-studied; in fact the matrices  $U_{kk}^\ell$ , for  $\ell = 1, \dots, k$  are the adjacency matrices of the Johnson scheme  $J(v, k)$  (see Section 5 for more details).

Other intersection matrices which are introduced in a different context are  $N_{sk}^t$  and  $A_{sk}^i$  defined by

$$\begin{aligned} N_{sk}^t(S, K) &= \binom{|S \cap K| - 1}{t}, \\ A_{sk}^i(S, K) &= \binom{|S \cap K|}{i}. \end{aligned}$$

The matrix  $N_{sk}^t$  was introduced in [9] (under the name  $M_{tk}^s$ ) and more discussed in [11] as an auxiliary tool to speed up an algorithmic search for finding  $t$ -designs with given parameters. (In that context the parameters are restricted so that  $t < k$  and  $t < s$ ; here we usually do not assume such restrictions on the parameters unless it is clear from the context or mentioned otherwise.) For the matrix  $A_{sk}^i$  one can observe that

$$A_{sk}^i = W_{is}^\top W_{ik}. \quad (4)$$

The intersection matrices  $W_{sk}$ ,  $\overline{W}_{sk}$ , and  $N_{sk}^t$  can be obtained from  $A_{sk}^i$ . In fact, (2) is rewritten

as

$$\overline{W}_{sk} = \sum_{i=0}^s (-1)^i A_{sk}^i, \quad (5)$$

and it is easily seen that

$$N_{sk}^t = \sum_{i=0}^t (-1)^{t-i} A_{sk}^i. \quad (6)$$

(We remark that  $\overline{W}_{sk} = (-1)^{\min(s,k)} N_{sk}^{\min(s,k)}$ .)

All the intersection matrices discussed above have appeared in different contexts in combinatorics including design theory, association scheme, extremal set theory, etc. The most significant properties of which making them a handy tool are the combinatorial identities they satisfy. The goal of the present paper is to introduce and investigate a more general framework in which the above intersection matrices arise as special cases and the identities involving them are derived more naturally. The structure of the paper is discussed in more details in the upcoming paragraphs.

In Section 2 an intersection matrix  $F_{sk}^t(v)(z)$  is introduced whose rows and columns are indexed by  $s$ -subsets and  $k$ -subsets of a  $v$ -set, respectively, and whose entries are polynomials in the indeterminate  $z$ . Indeed,  $F_{sk}^t(v)(z)$  can also be considered as a polynomial whose coefficients are some intersection matrices. This matrix is the most important tool to study other intersection matrices, their recursive structures and the identities satisfied by them. A new matrix  $U_{sk}^{t\ell}(v)$  is then defined as  $U_{sk}^{t\ell}(v) = \left. \frac{1}{\ell!} \frac{d^\ell}{dz^\ell} F_{sk}^t(v)(z) \right|_{z=-1}$ . This matrix generalizes all other matrices mentioned in this introduction (according to Theorem 2 (ii)). Another useful matrix studied here is  $F_{sk}(v)(z)$  defined as  $F_{sk}(v)(z) = F_{sk}^{\min(s,k)}(v)(z)$ .

Sections 3,4,5 and 6 are devoted to study some more facts about intersection matrices. In Section 3 we calculate the matrix product  $W_{is}^\top F_{ik}^t$  as a linear combination of derivations of  $F_{sk}^t$ . This reveals a close relation between this matrix product and the operation  $\frac{zd}{dz}$ . More precisely, we obtain  $W_{is}^\top F_{ik}^t = L_{si} F_{sk}^t$  where  $L_{si}$  is expressed as a polynomial in terms of  $\frac{zd}{dz}$ . The result is then helpful to calculate  $W_{sj} F_{jk}^t$  in Section 4 using equivalency of matrices. In Section 5 we obtain  $F_{sk}^t$  as a multiplication of a matrix  $X_{st}^k$  by  $W_{tk}$ . In Section 6 we demonstrate the block decomposition of all intersection matrices appeared in this paper.

In Section 7 two bases for the Bose-Mesner algebra of the Johnson scheme are studied using previous intersection matrices. In Section 8 we consider matrices  $F_{kk}^t(z)$  and  $W_{is}^\top F_{ik}^t$  as matrices with the entries in the field of rational function  $\mathbb{R}(z)$ , then obtain their eigenvalues. Also we compute the eigenvalues of matrices  $U_{kk}^{t\ell}$  and  $U_{kk}^{\geq \ell}$ . Moreover, we give a closed form for the eigenvalues and the rank of  $N_{kk}^{k-1}$ . The rank of  $U_{tk}^\ell$  is also investigated.

## 2 The matrices $F_{sk}^t(v)(z)$ and $U_{sk}^{t\ell}$

Equation (6) is a motivation to define the generation function  $F_{sk}^t(v)(z)$  as

$$F_{sk}^t(v)(z) = \sum_{i=0}^t A_{sk}^i(v) z^i. \quad (7)$$

When there is no danger of confusion, we drop  $v$  and/or  $z$  from the notation. The entry  $(S, K)$  of the above matrix is then computed as  $F_{sk}^t(v)(z)(S, K) = \psi_{\theta,t}(z)$  where  $\theta = |S \cap K|$  and

$$\psi_{\theta,t}(z) = \sum_{i=0}^t \binom{\theta}{i} z^i.$$

It is thus worthwhile to study the function  $\psi_{\theta,t}(z)$ . It is easily seen that:

$$\begin{aligned} \psi_{\theta,t+1} &= \psi_{\theta,t} + \binom{\theta}{t+1} z^{t+1}, \\ \psi_{\theta+1,t+1} &= \psi_{\theta,t+1} + z\psi_{\theta,t}, \\ \psi_{\theta+1,t+1} &= (z+1)\psi_{\theta,t} + \binom{\theta}{t+1} z^{t+1}, \end{aligned} \quad (8)$$

where the last equation is obtained from the two previous ones. Moreover, we have

$$D\psi_{\theta+1,t+1} = (\theta+1)\psi_{\theta,t}, \quad (9)$$

$$\psi_{\theta,t}(-1) = (-1)^t \binom{\theta-1}{t}, \quad (10)$$

where  $D$  in (9) means  $\frac{d}{dz}$  and the equation (10) is obtained using the identity  $\sum_{i=0}^t (-1)^i \binom{\theta}{i} = (-1)^t \binom{\theta-1}{t}$ .

According to the definition of  $\psi$ , if  $\theta \leq t$  then  $\psi_{\theta,t} = (z+1)^\theta$ . This encourages us to determine  $\psi_{\theta,t}$  in terms of  $z+1$  in general. Let  $\psi_{\theta,t}(z) = \sum_{\ell=0}^t a_\ell (z+1)^\ell$ . Then by (9) and (10)

$$\begin{aligned} a_\ell &= \frac{1}{\ell!} D^\ell \psi_{\theta,t}(z)_{z=-1} \\ &= \binom{\theta}{\ell} \psi_{\theta-\ell,t-\ell}(-1) \\ &= (-1)^{t-\ell} \binom{\theta}{\ell} \binom{\theta-\ell-1}{t-\ell}. \end{aligned} \quad (11)$$

This is a motivation to apply the same technique to the function  $F_{sk}^t$ . Write  $F_{sk}^t$  in terms of the powers of  $z + 1$  and let  $U_{sk}^{t\ell}(v)$  be the coefficient of  $(z + 1)^\ell$ , i.e.

$$F_{sk}^t(v)(z) = \sum_{\ell=0}^t U_{sk}^{t\ell}(v)(z + 1)^\ell. \quad (12)$$

From (7) and (12), for  $0 \leq i, \ell \leq t$ , we have

$$A_{sk}^i = \left. \frac{1}{i!} D^i F_{sk}^t(z) \right]_{z=0}, \quad (13)$$

$$U_{sk}^{t\ell} = \left. \frac{1}{\ell!} D^\ell F_{sk}^t(z) \right]_{z=-1}, \quad (14)$$

where  $D = \frac{d}{dz}$ . In the following we will see some results obtained by this technique. Applying (14) to (7) we obtain (15). This can simply be inverted as (16) by applying (13) to (12). This yields the following lemma:

**Proposition 1.** *The following identities hold to be true:*

$$U_{sk}^{t\ell} = \sum_{i=\ell}^t (-1)^{i-\ell} \binom{i}{\ell} A_{sk}^i, \quad (15)$$

$$A_{sk}^i = \sum_{\ell=i}^t \binom{\ell}{i} U_{sk}^{t\ell}. \quad (16)$$

In the following theorem we give a closed form for the entries of the matrix  $U_{sk}^{t\ell}$ . This concludes that the new matrix generalizes matrices  $U_{sk}^\ell$ ,  $N_{sk}^t$  and  $A_{sk}^t$ . The *support* of a vector is the number of nonzero elements in it. The support of each row of  $U_{sk}^{t\ell}$  is given in the following theorem. This is of interest from computational point of view.

**Theorem 2.** *Let  $t, \ell, s, k, v$  be integers with  $\ell \leq t$  and  $s, k \leq v$ , and let  $B = \{\ell\} \cup \{t + 1, \dots, \min(s, k)\}$ .*

(i) *For every  $s$ -subset  $S$  and  $k$ -subset  $K$  of  $\{1, \dots, v\}$  with  $\theta = |S \cap K|$ ,*

$$U_{sk}^{t\ell}(S, K) = (-1)^{t-\ell} \binom{\theta}{\ell} \binom{\theta - \ell - 1}{t - \ell}$$

*and this value is nonzero only if  $\theta \in B$ . If moreover  $\ell \leq t \leq \min(s, k)$ , then the support of each row of  $U$  is  $\sum_{\theta \in B} \binom{s}{\theta} \binom{v-s}{k-\theta}$ .*

(ii)  $U_{sk}^{t,0} = (-1)^t N_{sk}^t$ ,  $U_{sk}^{s\ell} = U_{sk}^\ell$ , and  $U_{sk}^{tt} = A_{sk}^t$ .

(iii)

$$\begin{aligned} U_{sk}^{t\ell} &= \sum_{\theta \in B} (-1)^{t-\ell} \binom{\theta}{\ell} \binom{\theta-\ell-1}{t-\ell} U_{sk}^{\theta} \\ &= U_{sk}^{\ell} + \sum_{t+1 \leq \theta \leq \min(s,k)} (-1)^{t-\ell} \binom{\theta}{\ell} \binom{\theta-\ell-1}{t-\ell} U_{sk}^{\theta}. \end{aligned}$$

**Proof.** To prove (i), note that  $U_{sk}^{t\ell}(S, K)$  is computed in a similar way as  $a_{\ell}$  is done in the beginning of Section 2. Moreover, it is easy to see that this value is nonzero if and only if either  $\theta = \ell$  or  $t+1 \leq \theta \leq \min(s, k)$ . These prove (i). (ii) is an immediate consequence of (i). (iii) follows from (i) and (ii).  $\square$

The matrices  $N_{7,7}^6(14) = U_{7,7}^{6,0}(14)$  and  $N_{6,6}^5(13) = -U_{6,6}^{5,0}(13)$  had important roles in finding  $t$ -designs with related parameters (see [9]). We consider these matrices in the following examples. These are discussed more in feature sections.

**Example 1.** The matrix  $N_{7,7}^6(14)$  is a  $(0, 1)$ -matrix in which  $N_{7,7}^6(14)(K_1, K_2) = 1$  if and only if either  $K_1 = K_2$  or  $K_1 \cap K_2 = \emptyset$ . The support of each row of this matrix is 2. Note that the rows of this matrix are included in the row-space of  $W_{6,7}(14)$  and have minimum support among all vectors in this space.

**Example 2.** The matrix  $N_{6,6}^5(13)$  is a  $(0, \pm 1)$ -matrix in which  $N_{6,6}^5(13)(K_1, K_2)$  equals  $+1$  ( $-1$ ) if  $K_1 = K_2$  ( $K_1 \cap K_2 = \emptyset$ ). The support of each row of this matrix is 8 which is the same as the support of each row of  $W_{5,6}(13)$ .

Define  $F_{sk} := F_{sk}^{\min(s,k)}$ . Then  $F_{sk}(S, K) = \sum_{\ell=0}^s (z+1)^{\ell} U_{sk}^{\ell}(S, K) = \sum_{\ell=0}^s (z+1)^{\ell} \delta_{|S \cap K|, \ell}$ . Hence

$$F_{sk}(S, K) = (z+1)^{|S \cap K|}.$$

**Lemma 3.**

- (i)  $(F_{sk}^t)^{\top}(z) = F_{ks}^t(z)$ . Hence,  $F_{kk}^t(z)$  is a symmetric matrix.
- (ii) If  $t \geq \min(s, k)$ , then  $F_{sk}^t(z) = F_{sk}(z)$ .
- (iii)  $F_{kk}(z)F_{kk}(u) = F_{kk}(u)F_{kk}(z)$ .
- (iv)  $A_{kk}^i A_{kk}^j = A_{kk}^j A_{kk}^i$ .
- (v)  $U_{kk}^i U_{kk}^j = U_{kk}^j U_{kk}^i$ .

**Proof.** (i) This is an immediate consequence of the definition of  $F_{sk}^t$ .

(ii) We have  $F_{sk}^t(S, K) = \psi_{\theta, t}$  where  $\theta = |S \cap K|$ . Note that  $\theta \leq \min(s, k) = t$  so  $\psi_{\theta, t} = (z+1)^\theta$  and  $F_{sk}^t = F_{sk}$ .

(iii) Let  $K_1, K_2 \in P_k(v)$ . Then since  $K_1 \setminus K_2$  and  $K_2 \setminus K_1$  are disjoint subsets of  $V$  with the same cardinality, there exists a permutation  $\sigma \in S_v$  which maps  $K_1 \setminus K_2$  to  $K_2 \setminus K_1$  and vice versa and fixes the other points. We then have

$$\begin{aligned} (F_{kk}(z)F_{kk}(u))(K_1, K_2) &= \sum_{K \in P_k(v)} (z+1)^{|K_1 \cap K|} (u+1)^{|K_2 \cap K|} \\ &= \sum_{K \in P_k(v)} (z+1)^{|K_1 \cap \sigma(K)|} (u+1)^{|K_2 \cap \sigma(K)|} \\ &= \sum_{K \in P_k(v)} (z+1)^{|K_2 \cap K|} (u+1)^{|K_1 \cap K|} \\ &= (F_{kk}(u)F_{kk}(z))(K_1, K_2). \end{aligned}$$

(iv),(v) These are concluded from (iii).

□

For two matrices  $A$  and  $B$ , we write  $A \sim B$  if  $A$  can be obtained from  $B$  by a permutation on rows and a permutation on columns. In this case, we say that matrices  $A$  and  $B$  are equivalent. For instance  $\overline{W}_{sk} \sim W_{s, v-k}$ . More generally, it is easily seen that

$$U_{sk}^\ell \sim U_{s, v-k}^{s-\ell} \sim U_{v-s, k}^{k-\ell} \sim U_{v-s, v-k}^{v-s-k+\ell}.$$

In the following, similarity of matrices is used to find a proof for the identity useful identity (18), given in [12, 13], based on the definition of  $F_{sk}(v)(z)$ . The identities following (18) in this proposition are immediate results of it. These are used in future sections. We use the notation  $[z^i]F$  to denote the coefficient of  $z^i$  in  $F$ .

**Proposition 4.**

$$F_{v-a, v-b}(z) \sim (z+1)^{v-a-b} F_{a, b}(z) \quad (17)$$

$$W_{ak} W_{bk}^\top = \sum_{n=0}^{\min(a, b)} \binom{v-b-a}{v-k-n} A_{ab}^n \quad (18)$$

$$A_{ab}^i A_{bc}^j = \sum_{n=0}^{\min(i, j)} \binom{a-n}{i-n} \binom{c-n}{j-n} \binom{v-i-j}{b+n-i-j} A_{ac}^n \quad (19)$$

$$U_{ab}^i U_{bc}^j = \sum_{\ell=0}^{\min(a, c)} \sum_{n=0}^{\ell} \binom{\ell}{n} \binom{c-\ell}{j-n} \binom{a-\ell}{i-n} \binom{v-a-c+\ell}{b-i-j+n} U_{ac}^\ell \quad (20)$$

**Proof.** Let  $A' = V \setminus A$  and  $B' = V \setminus B$ . Then

$$\begin{aligned} F_{v-a, v-b}(A', B') &= (z+1)^{|A' \cap B'|} \\ &= (z+1)^{v-a-b+|A \cap B|} \\ &= (z+1)^{v-a-b} F_{ab}(A, B). \end{aligned}$$

This proves (17). Note that  $W_{a,k} \sim W_{v-k, v-a}^\top$  because  $A \subseteq K$  if and only if  $V \setminus K \subseteq V \setminus A$ . If one applies simultaneously a same permutation on the columns of  $W_{ak}$  and the rows of  $W_{bk}^\top$ ,  $W_{a,k} W_{b,k}^\top \sim W_{v-k, v-a}^\top W_{v-k, v-b}$ . Whence

$$\begin{aligned} W_{a,k} W_{b,k}^\top &\sim A_{v-a, v-b}^{v-k} \\ &= [z^{v-k}] F_{v-a, v-b} \\ &\sim [z^{v-k}] ((z+1)^{v-a-b} F_{ab}) \\ &= \sum_{j=0}^{\min(a,b)} \binom{v-a-b}{v-k-j} [z^j] F_{ab} \\ &= \sum_{j=0}^{\min(a,b)} \binom{v-a-b}{v-k-j} A_{ab}^j, \end{aligned}$$

concluding (18). (We note that the same ordering is used for the rows and the columns of both matrices.) To prove (19), replace  $a$  and  $k$  in (18) respectively by  $i$  and  $j$ . Then multiply the identity from left and right respectively by  $W_{ia}^\top$  and  $W_{jc}$  and use (1).

To prove (20), let  $A$  and  $C$  be respectively an  $a$ -subset and a  $c$ -subset of  $\{1, \dots, v\}$ . It is easily seen that

$$(U_{ab}^i U_{bc}^j)(A, C) = |\{B \subset \{1, \dots, v\} : |B| = b, |B \cap A| = i, |B \cap C| = j\}|.$$

Let  $\ell = |A \cap C|$  and  $n = |A \cap B \cap C|$ . To construct  $B$  one should select  $n$  points out of  $A \cap C$ ,  $i - n$  points out of  $A \setminus C$ ,  $j - n$  points out of  $C \setminus A$  and  $b - i - j + n$  points out of  $A' \cap C'$ . Thus we obtain

$$(U_{ab}^i U_{bc}^j)(A, C) = \sum_{\ell=0}^{\min(a,c)} \delta_{|A \cap C|, \ell} \sum_{n=0}^{\ell} \binom{\ell}{n} \binom{c-\ell}{j-n} \binom{a-\ell}{i-n} \binom{v-a-c+\ell}{b-i-j+n}$$

which provides the result □

### 3 Calculating $W_{is}^\top F_{ik}^t$

In this section we obtain a formula for the matrix product  $W_{sj}^\top F_{jk}^t$ . We mention that using the definition of  $F$  and previously mentioned equation one can compute this matrix product. However,



we use another method which reveals the relationship between the operator  $zD$  and this matrix product and gives the result as an expression in terms of derivations of  $F$  as a natural consequence. This equality can be considered as a differential equation containing matrices. We remark that some recursive equations satisfied by intersection numbers are previously studied in another framework (See for instance [4]).

Below some definitions and facts used in this section are listed.

We frequently make use of the following identity which holds for nonnegative integers  $\ell, m, n$ .

$$\sum_{k \leq \ell} (-1)^k \binom{\ell - k}{m} \binom{s}{k - n} = (-1)^{\ell + m} \binom{s - m - 1}{\ell - m - n} \quad (21)$$

Following [10] we use the *falling factorial* notation

$$(x)_n := x(x - 1) \cdots (x - n + 1).$$

We recall that the *Stirling numbers of the second kind*, denoted by  $S(n, k)$ , are the numbers of ways of partitioning an  $n$ -set into exactly  $k$  parts; and the *Stirling numbers of the first kind* are the number of permutations of  $S_n$  having exactly  $k$  cycles. It is well-known that the following recursive identities hold for  $1 < k \leq n$ :

$$\begin{aligned} S(n + 1, k) &= kS(n, k) + S(n, k - 1) \\ s(n + 1, k) &= -ns(n, k) + s(n, k - 1) \end{aligned}$$

Moreover it is well-known that

$$\begin{aligned} \sum_{k=1}^n s(n, k) x^k &= (x)_n \\ \sum_{k=1}^n S(n, k) x^k &= x^n \\ \sum_{k=m}^n s(n, k) S(k, m) &= \delta_{n, m}. \end{aligned}$$

**Proposition 5.**

- (i)  $W_{s-1, s}^\top F_{s-1, k}^t(z) = sF_{s, k}^t(z) - zDF_{s, k}^t(z),$
- (i)'  $F_{s, k-1}^t(z)W_{k-1, k} = kF_{s, k}^t(z) - zDF_{s, k}^t(z),$
- (ii)  $W_{s-1, s}^\top F_{s-1, k}(z) = sF_{s, k}(z) - zDF_{s, k}(z),$
- (ii)'  $F_{s, k-1}(z)W_{k-1, k} = kF_{s, k}(z) - zDF_{s, k}(z),$
- (iii)  $W_{s-1, s}^\top U_{s-1, k}^{t, \ell} = (s - \ell)U_{s, k}^{t, \ell} + (\ell + 1)U_{s, k}^{t, \ell+1},$

$$(iii)' \quad U_{s,k-1}^{t,\ell} W_{k-1,k} = (k-\ell)U_{s,k}^{t,\ell} + (\ell+1)U_{s,k}^{t,\ell+1},$$

$$(iv) \quad W_{s-1,s}^\top U_{s-1,k}^\ell = (s-\ell)U_{s,k}^\ell + (\ell+1)U_{s,k}^{\ell+1},$$

$$(iv)' \quad U_{s,k-1}^\ell W_{k-1,k} = (k-\ell)U_{s,k}^\ell + (\ell+1)U_{s,k}^{\ell+1},$$

**Proof.** (i),(i)' By (4) and (1) we have  $W_{s-1,s}^\top A_{s-1,k}^i = (s-i)A_{s,k}^i$ . Therefore

$$\begin{aligned} W_{s-1,s}^\top F_{s-1,k}^t(z) &= \sum_{i=0}^t (s-i)A_{s,k}^i z^i \\ &= s \sum_{i=0}^t A_{s,k}^i z^i - z \sum_{i=1}^t A_{s,k}^i i z^{i-1}, \end{aligned}$$

which proves (i). (i)' follows from (i) and Lemma 3(i).

(ii),(ii)' Let  $t = s$  in the equations of parts (i),(i)'.

(iii),(iii)' By applying the operator  $D^\ell$  to (i) we get

$$W_{s-1,s}^\top D^\ell F_{s-1,k}^t = (s-\ell)D^\ell F_{s,k}^t - zD^{\ell+1}F_{s,k}^t.$$

Then using (14) we obtain (iii). The other one is proved similarly.

(iv),(iv)' Let  $t = s$  in (iii),(iii)'.

□

In part (iii) of the previous proposition the expression  $W_{i,s}^\top U_{i,k}^{t,\ell}$  is calculated for  $i = s-1$ , but how can we calculate this expression in general? When  $t = k$  the answer of this question is obtained through a simple counting argument (this argument is appeared in the proof of part (ii)' of Proposition 7). To find the answer in general, first notice that Proposition 5(i) can be written as  $W_{s-1,s}^\top F_{s-1,k}^t(z) = (s-zD)F_{s,k}^t(z)$  (Here  $s-zD$  stands for  $s\mathbf{1} - zD$  where  $\mathbf{1}$  is the identity operator). Next, note that from (1) it follows that

$$W_{i,i+1}W_{i+1,i+2} \cdots W_{s-1,s} = (s-i)!W_{is}.$$

Now by iterative use of proposition 5 (i) we obtain

$$\begin{aligned} W_{is}^\top F_{ik}^t &= \frac{1}{(s-i)!} W_{s-1,s}^\top W_{s-2,s-1}^\top \cdots W_{i,i+1}^\top F_{ik}^t \\ &= \frac{1}{(s-i)!} (s-zD)(s-1-zD) \cdots (i+1-zD) F_{sk}^t \end{aligned}$$

With slight change of notation, we can write more briefly

$$W_{is}^\top F_{ik}^t = L_{si} F_{sk}^t,$$

where

$$\begin{aligned} L_{si} &= \frac{1}{(s-i)!} (s-zD)(s-1-zD) \cdots (i+1-zD) \\ &= \frac{(-1)^{s-i}}{(s-i)!} (zD-i-1)_{s-i} \end{aligned} \quad (22)$$

To simplify  $L_{si}$ , first we should study the operator  $zD$ . This is done in the following lemma.

**Lemma 6.** *Let  $n$  be a positive integer. Then*

- (i)  $(zD)^n = \sum_{k=1}^n S(n, k) z^k D^k$ ,
- (ii)  $(zD)_n = z^n D^n$ ,
- (iii)  $(zD-k)_n = n! \sum_{r=0}^n \binom{n+k-r-1}{n-r} (-1)^{n-r} \frac{z^r}{r!} D^r$ .

**Proof.** (i) The proof is by induction on  $n$  using the identity  $S(n+1, k) = kS(n, k) + S(n, k-1)$ .

(ii) By the definition of  $s(n, k)$ ,

$$\begin{aligned} (zD)_n &= \sum_{k=1}^n s(n, k) (zD)^k \\ &= \sum_{k=1}^n s(n, k) \sum_{i=1}^k S(k, i) z^i D^i \\ &= \sum_{k=1}^n \sum_{i=1}^k s(n, k) S(k, i) z^i D^i \\ &= \sum_{i=1}^n \left( \sum_{k=i}^n s(n, k) S(k, i) \right) z^i D^i \\ &= \sum_{i=1}^n \delta_{n,i} z^i D^i \\ &= z^n D^n. \end{aligned}$$

(iii) Using the identities

$$(x+k)_n = \sum_{r=0}^n \binom{n}{r} (x)_r (k)_{n-r}, \text{ and } (-k)_i = (-1)^i (k+i-1)_i,$$

we have

$$\begin{aligned}
(zD - k)_n &= \sum_{r=0}^n \binom{n}{r} (zD)_r (-k)_{n-r} \\
&= \sum_{r=0}^n \binom{n}{r} (-1)^{n-r} (n+k-r-1)_{n-r} (zD)_r \\
&= \sum_{r=0}^n \binom{n}{r} (-1)^{n-r} (n+k-r-1)_{n-r} z^r D^r \\
&= n! \sum_{r=0}^n \binom{n+k-r-1}{n-r} (-1)^{n-r} \frac{z^r}{r!} D^r,
\end{aligned}$$

which implies the result.  $\square$

**Proposition 7.**

$$(i) \quad W_{is}^\top F_{ik}^t = \sum_{r=0}^{s-i} (-1)^r \binom{s-r}{i} \frac{z^r}{r!} D^r F_{sk}^t,$$

$$(i') \quad W_{is}^\top F_{ik}^t = \sum_{r=0}^{s-i} (-1)^r \binom{s-r}{i} \frac{z^r}{r!} D^r F_{sk}^t,$$

$$(ii) \quad W_{is}^\top U_{ik}^{t\ell} = \sum_{h=\ell}^{\ell+s-i} \binom{h}{\ell} \binom{s-h}{i-\ell} U_{sk}^{th}.$$

$$(ii') \quad W_{is}^\top U_{ik}^\ell = \sum_{h=\ell}^s \binom{h}{\ell} \binom{s-h}{i-\ell} U_{sk}^h$$

**Proof.** (i) We have  $W_{is}^\top F_{ik}^t = L_{si} F_{sk}^t$  and  $L_{si}$  can be expanded according to Lemma 6 as follows

$$\begin{aligned}
L_{si} &= \frac{(-1)^{s-i}}{(s-i)!} (zD - i - 1)_{s-i} \\
&= \frac{(-1)^{s-i}}{(s-i)!} (s-i)! \sum_{r=0}^{s-i} \binom{s-r}{s-i-r} (-1)^{s-i-r} \frac{z^r}{r!} D^r \\
&= \sum_{r=0}^{s-i} (-1)^r \binom{s-r}{i} \frac{z^r}{r!} D^r.
\end{aligned}$$

(i') Considering Lemma 3(ii), the result is obtained by setting  $t = s$  in part (i).

(ii) Applying the operator  $D^\ell/\ell!$  on (i),

$$W_{is}^\top \frac{D^\ell}{\ell!} F_{ik}^t = \frac{1}{\ell!} \sum_{r=0}^{s-i} \frac{(-1)^r}{r!} \binom{s-r}{i} D^\ell (z^r D^r) F_{sk}^t.$$

On the other hand, by Leibniz's rule

$$D^\ell(z^r D^r) = \sum_{j=0}^{\ell} \binom{\ell}{j} (r)_{\ell-j} z^{r-\ell+j} D^{r+j}.$$

Thus,

$$W_{is}^\top \frac{D^\ell}{\ell!} F_{ik}^t(z) = \frac{1}{\ell!} \sum_{r=0}^{s-i} \frac{(-1)^r}{r!} \binom{s-r}{i} \sum_{j=0}^{\ell} \binom{\ell}{j} (r)_{\ell-j} z^{r-\ell+j} D^{r+j} F_{sk}^t(z).$$

Letting  $z = -1$  in the above we get

$$\begin{aligned} W_{is}^\top U_{ik}^{t\ell} &= \sum_{r=0}^{s-i} \sum_{j=0}^{\ell} \binom{s-r}{i} \binom{r+j}{j} \binom{r}{\ell-j} (-1)^{\ell-j} U_{sk}^{t, r+j} \\ &= \sum_{h=0}^{\ell+s-i} a_h U_{sk}^{th}, \end{aligned}$$

where

$$\begin{aligned} a_h &= \sum_{j=0}^h (-1)^{\ell-j} \binom{h}{j} \binom{h-j}{\ell-j} \binom{s-h+j}{i} \\ &= \binom{h}{\ell} \sum_{j=0}^h (-1)^{\ell-j} \binom{\ell}{j} \binom{s-h+j}{i} \\ &= \binom{h}{\ell} (-1)^{\ell-h} \sum_{j=0}^h (-1)^{h-j} \binom{\ell}{\ell-j} \binom{s-(h-j)}{i} \\ &= \binom{h}{\ell} (-1)^{\ell-h} \sum_{j=0}^h (-1)^j \binom{\ell}{\ell-h+j} \binom{s-j}{i} \\ &= (-1)^{\ell-h} \binom{h}{\ell} (-1)^{s+i} \binom{\ell-i-1}{s-i-h+\ell} \\ &= \binom{h}{\ell} \binom{s-h}{s-i-h+\ell} \\ &= \binom{h}{\ell} \binom{s-h}{i-\ell}. \end{aligned}$$

(ii') Considering Lemma 3(ii), the result is obtained by setting  $t = s$  in part (ii). We give an alternative direct proof for this identity by calculating the  $(S, K)$  entry of the matrices. It is observed that

$$(W_{is}^\top U_{ik}^\ell)(S, K) = |\{I : |I| = i, I \subseteq S, |I \cap K| = \ell\}|.$$

Let  $P = |S \cap K|$ ,  $Q = |S \setminus K|$ . If  $|P| = h$  then  $|I \cap P| = \ell$ ,  $|I \cap Q| = i - \ell$ , and  $|\{I : |I| = i, I \subseteq S, |I \cap K| = \ell\}| = \binom{h}{\ell} \binom{s-h}{i-\ell}$ . The right side, thus, equals

$$\sum_h \binom{h}{\ell} \binom{s-h}{i-\ell} \delta_{|S \cap K|, h} = \sum_h \binom{h}{\ell} \binom{s-h}{i-\ell} U_{sk}^h(S, K),$$

which gives the result.  $\square$

Note that by transposing the equalities of Proposition 5, one can obtain the values of  $F_{sj}^t W_{jk}$ ,  $F_{sj} W_{jk}$ ,  $U_{sj}^{t\ell} W_{jk}$  and  $U_{sj}^\ell W_{jk}$ .

## 4 Calculating $W_{sj} F_{jk}$

In this section we find the matrix product  $W_{sj} F_{jk}$ . We use equivalency of matrices to deduce directly the above matrix product from the results of the previous section.

**Proposition 8.** *The following identity holds*

$$W_{sj} F_{jk} = (z+1)^{-v+j+k} \sum_{r=0}^{j-s} (-1)^r \binom{v-s-r}{v-j} \frac{z^r D^r}{r!} ((z+1)^{v-s-k} F_{sk}) \quad (23)$$

Hence

$$W_{sj} F_{jk} = \sum_{p=0}^{j-s} \frac{(z+1)^p D^p (F_{sk})}{p!} \sum_{\ell=0}^{j-s} (-1)^\ell a_{p,\ell} (z+1)^{j-s-\ell}, \quad (24)$$

where  $a_{p,\ell}$  is defined as

$$a_{p,\ell} = \sum_{r=0}^{j-s} \binom{r}{\ell} \binom{v-s-r}{v-j} \binom{v-s-k}{r-p}. \quad (25)$$

**Proof.** The main idea of the proof is similar to the one used to prove identity (18) of proposition 4. Note that  $W_{s,j} \sim W_{v-j,v-s}^\top$  and  $F_{jk} \sim (z+1)^{-v+j+k} F_{v-j,v-k}$  and because of the proper orderings of rows and columns we obtain

$$\begin{aligned} W_{sj} F_{jk} &\sim (z+1)^{-v+j+k} W_{v-j,v-s}^\top F_{v-j,v-k} \\ &= (z+1)^{-v+j+k} \sum_{r=0}^{j-s} (-1)^r \binom{v-s-r}{v-j} \frac{z^r D^r}{r!} (F_{v-s,v-k}) \\ &\sim (z+1)^{-v+j+k} \sum_{r=0}^{j-s} (-1)^r \binom{v-s-r}{v-j} \frac{z^r D^r}{r!} ((z+1)^{v-s-k} F_{sk}) \end{aligned}$$

Since the rows (columns) of the last matrix has the same ordering as the rows (columns) of  $W_{sj} F_{jk}$  equality holds. Hence

$$\begin{aligned}
W_{sj}F_{jk} &= (z+1)^{-v+j+k} \sum_{r=0}^{j-s} (-1)^r \binom{v-s-r}{v-j} \frac{z^r D^r}{r!} ((z+1)^{v-s-k} F_{sk}) \\
&= (z+1)^{-v+j+k} \sum_{r=0}^{j-s} (-1)^r \binom{v-s-r}{v-j} \frac{z^r}{r!} \sum_{p=0}^r \binom{r}{p} (v-s-k)_{r-p} (z+1)^{v-s-k-r+p} D^p F_{sk} \\
&= (z+1)^{-v+j+k} \sum_{r=0}^{j-s} \sum_{p=0}^r (-1)^r \binom{v-s-r}{v-j} z^r \binom{v-s-k}{r-p} (z+1)^{v-s-k-r+p} \frac{D^p F_{sk}}{p!} \\
&= (z+1)^{-v+j+k} \sum_{p=0}^{j-s} \sum_{r=p}^{j-s} \left(\frac{-z}{z+1}\right)^r \binom{v-s-r}{v-j} \binom{v-s-k}{r-p} (z+1)^{v-s-k+p} \frac{D^p F_{sk}}{p!} \\
&= (z+1)^{j-s} \sum_{p=0}^{j-s} (z+1)^p \frac{D^p F_{sk}}{p!} \sum_{r=p}^{j-s} \binom{v-s-r}{v-j} \binom{v-s-k}{r-p} (z+1)^{v-s-k+p} \left(\frac{-z}{z+1}\right)^r \\
&= (z+1)^{j-s} \sum_{p=0}^{j-s} (z+1)^p \frac{D^p F_{sk}}{p!} \sum_{r=p}^{j-s} \binom{v-s-r}{v-j} \binom{v-s-k}{r-p} (z+1)^{v-s-k+p} \left(-1 + \frac{1}{z+1}\right)^r \\
&= \sum_{p=0}^{j-s} (z+1)^p \frac{D^p F_{sk}}{p!} \sum_{\ell=0}^{j-s} (-1)^\ell a_{p,\ell} (z+1)^{j-s-\ell}
\end{aligned}$$

where  $a_{p,\ell}$  is as defined in the lemma. □

## 5 Factoring out $W_{tk}$

In this section we factor out  $W_{tk}$  from  $F_{sk}^t$  and study the resulting matrix.

By Equations (7), (4) and (1),

$$F_{sk}^t = X_{st}^k(z) W_{tk}, \quad \text{where} \quad X_{st}^k(z) = \sum_{i=0}^t \frac{1}{\binom{k-i}{t-i}} A_{st}^i z^i. \quad (26)$$

Hence  $X_{st}^k(z)(S, T) = \xi_{\theta,t}^k(z)$  where  $\theta = |S \cap T|$  and the function  $\xi$  is defined as

$$\xi_{\theta,t}^k(z) = \sum_{i=0}^t \frac{\binom{\theta}{i}}{\binom{k-i}{t-i}} z^i = \sum_{i=0}^{\theta} \frac{\binom{\theta}{i}}{\binom{k-i}{t-i}} z^i. \quad (27)$$

Note that in the definition of  $\xi$  the upper bound of the summation is  $\theta$  instead of  $t$ . It is now worthwhile to study the function  $\xi_{\theta,t}^k$ . This function satisfies the following recursive equations:

$$\xi_{\theta+1,t+1}^{k+1} = \xi_{\theta,t+1}^{k+1} + z \xi_{\theta,t}^k, \quad (28)$$

$$D \xi_{\theta+1,t+1}^{k+1} = \theta \xi_{\theta,t}^k. \quad (29)$$

In the following lemma the value of  $\xi_{\theta,t}^k(-1)$  is calculated. To simplify the obtained summation, we use high-order differences method (This method is discussed for instance in Sections 2.6 and 5.3 of [7]).

**Lemma 9.** *Let  $(k-t, \theta) \neq (0, 0)$ . Then*

$$\xi_{\theta,t}^k(-1) = (-1)^\theta \frac{k-t}{(k-t+\theta) \binom{k}{t-\theta}}. \quad (30)$$

**Proof.** By the definition of  $\xi$ ,  $\xi_{\theta,t}^k(-1) = \sum_{i=0}^{\theta} (-1)^i \binom{\theta}{i} f_{i,t}$ , where  $f_{i,t} = 1/\binom{k-i}{t-i}$ . Defining the operator  $\Delta$  by  $\Delta f_{i,t} = f_{i+1,t} - f_{i,t}$ ,

$$\xi_{\theta,t}^k(-1) = (-1)^\theta \Delta^\theta f_{j,t} \Big|_{j=0}.$$

On the other hand, it is easily proved (by induction on  $m$ ) that for any nonnegative integer  $m \leq t-j$ ,

$$\Delta^m f_{j,t} = \frac{k-t}{k-t+m} f_{j,t-m},$$

which completes the proof.  $\square$

Note that  $\xi_{\theta,t}^t = (z+1)^\theta$  and  $\xi_{0,t}^t = 1$ . This is also follows from (30) if we simply eliminate the term  $k-t$  from nominator and denominator in the case  $\theta = 0$ . With this convention, the condition  $(k-t, \theta) \neq (0, 0)$  is not necessary in the above lemma.

**Proposition 10.** *Let  $Y_{st}^{k\ell}$  be the matrix with the entries*

$$Y_{st}^{k\ell}(S, T) = \frac{(-1)^{\theta-\ell} \binom{\theta}{\ell} (k-t)}{(k-t+\theta-\ell) \binom{k-\ell}{t-\theta}}. \quad (31)$$

Then

$$U_{sk}^{t\ell} = Y_{st}^{k\ell} W_{tk}.$$

**Proof.** Let  $\xi_{\theta,t}^k = \sum_{\ell=0}^{\theta} b_\ell (z+1)^\ell$ . Then

$$\begin{aligned} b_\ell &= \left. \frac{D^\ell}{\ell!} \xi_{\theta,t}^k(z) \right|_{z=-1} \\ &= \binom{\theta}{\ell} \xi_{\theta-\ell,t-\ell}^{k-\ell}(-1) \quad (\text{by (29)}) \\ &= \frac{(-1)^{\theta-\ell} \binom{\theta}{\ell} (k-\ell)}{(k-t+\theta-\ell) \binom{k-\ell}{t-\theta}} \quad (\text{by (30)}) . \end{aligned}$$

It turns out that  $X_{st}^k(z) = \sum_{\ell=0}^t Y_{st}^{k\ell} (z+1)^\ell$ . Comparing (12) and (26) completes the proof.  $\square$



## 6 Block decompositions

It is well-known that the matrix  $W_{sk}(v)$  has the following recursive structure:

$$W_{sk}(v) = \begin{array}{|c|c|} \hline W_{s-1,k-1}(v-1) & O \\ \hline W_{s,k-1}(v-1) & W_{sk}(v-1) \\ \hline \end{array}.$$

This block decomposition played a significant role in determining a diagonal form for  $W_{sk}(v)$  as well as in studying integral solutions of the system  $W_{sk}\mathbf{x} = \mathbf{b}$  ( We mention that the integral solutions of  $W_{tk}\mathbf{x} = \mathbf{b}$  are called ‘signed  $t$ -designs’), see [6, 15].

In this section, we study a recursive decomposition of  $F_{sk}^t(v)(z)$ . We obtain general results which in turn induce decompositions on  $U_{sk}^{t\ell}$ .

**Theorem 11.** *We have the following recursive structures:*

$$(i) \quad F_{s,k}^t(v)(z) = \begin{array}{|c|c|} \hline (z+1)F_{s-1,k-1}^{t-1}(v-1)(z) + z^t A_{s-1,k-1}^t(v-1) & F_{s-1,k}^t(v-1)(z) \\ \hline F_{s,k-1}^t(v-1)(z) & F_{s,k}^t(v-1)(z) \\ \hline \end{array},$$

$$(ii) \quad F_{s,k}(v)(z) = \begin{array}{|c|c|} \hline (z+1)F_{s-1,k-1}(v-1)(z) & F_{s-1,k}(v-1)(z) \\ \hline F_{s,k-1}(v-1)(z) & F_{s,k}(v-1)(z) \\ \hline \end{array},$$

$$(iii) \quad U_{s,k}^{t,\ell}(v) = \begin{array}{|c|c|} \hline U_{s-1,k-1}^{t-1,\ell-1}(v-1) + (-1)^{t-\ell} \binom{t}{\ell} A_{s-1,k-1}^t(v-1) & U_{s-1,k}^{t,\ell}(v-1) \\ \hline U_{s,k-1}^{t,\ell}(v-1) & U_{s,k}^{t,\ell}(v-1) \\ \hline \end{array},$$

$$(iv) \quad U_{s,k}^\ell(v) = \begin{array}{|c|c|} \hline U_{s-1,k-1}^{\ell-1}(v-1) & U_{s-1,k}^\ell(v-1) \\ \hline U_{s,k-1}^\ell(v-1) & U_{s,k}^\ell(v-1) \\ \hline \end{array},$$

$$(v) \quad N_{s,k}^t(v) = \begin{array}{|c|c|} \hline A_{s-1,k-1}^t(v-1) & N_{s-1,k}^t(v-1) \\ \hline N_{s,k-1}^t(v-1) & N_{s,k}^t(v-1) \\ \hline \end{array},$$

$$(vi) \quad A_{s,k}^t(v) = \begin{array}{|c|c|} \hline A_{s-1,k-1}^t(v-1) + A_{s-1,k-1}^{t-1}(v-1) & A_{s-1,k}^t(v-1) \\ \hline A_{s,k-1}^t(v-1) & A_{s,k}^t(v-1) \\ \hline \end{array}.$$

**Proof.** (i) Clearly the first  $\binom{v-1}{s-1}$   $s$ -subsets of  $\{1, \dots, v\}$  in the lexicographic ordering contain the element 1 and the rest of them do not contain 1. Using the similar fact about the columns, the matrix  $F$  is divided into four blocks. Now we determine the four blocks. We do this for the block (1,1). Let  $1 \in S$  and  $1 \in K$  and let  $S' = S \setminus \{1\}$  and  $K' = K \setminus \{1\}$  and  $\theta = |S \cap K|$  and  $\theta' = |S' \cap K'|$ . Then  $\theta = \theta' + 1$  and

$$\begin{aligned} F_{sk}^t(S, K) &= \psi_{\theta'+1,t} \\ &= (z+1)\psi_{\theta',t-1} + \binom{\theta'}{t} z^t \quad [\text{by equation (8)}] . \end{aligned}$$

Thus the block (1,1) equals  $(z+1)F_{s-1,k-1}^{t-1}(v-1) + z^t A_{s-1,k-1}^t(v-1)$ . The other blocks are

determined similarly.

(ii) Let  $t = s$  in part (i).

(iii) Apply the operator  $D^\ell/\ell!$  to the both sides of part (i) and set  $z = -1$ .

(iv) Let  $s = t$  in part (iii).

(v) This is obtained from (iii) by setting  $t = 0$ .

(vi) This is obtained from (ii) using  $A_{sk}^t = \frac{D^t}{t!}F_{sk}(0)$ . □

## 7 Johnson scheme

An *association scheme with  $d$  classes* is a set of  $d + 1$  square  $(0, 1)$ -matrices  $X_0, X_1, \dots, X_d$  which satisfy

$$(i) \quad \sum_{i=0}^d X_i = J,$$

$$(ii) \quad X_0 = I,$$

$$(iii) \quad X_i = X_i^\top, \text{ for } i = 0, 1, \dots, d,$$

$$(iv) \quad X_i X_j = \sum_{\ell=0}^d p_{ij}^\ell X_\ell, \text{ for } i, j \in \{0, 1, \dots, d\}.$$

The numbers  $p_{ij}^\ell$  are called the *intersection numbers* of the association scheme. From (i) we see that the matrices  $X_i$  are linearly independent, and by use of (ii)–(iv) we see that they generate a commutative  $(d + 1)$ -dimensional algebra of symmetric matrices with constant diagonal. This algebra is called the *Bose-Mesner algebra* of the association scheme.

The *Johnson scheme*  $J(v, k)$  is a  $k$ -class association scheme in which the rows and the columns of each  $X_i$  is indexed by all  $k$ -subsets of a  $v$ -set and  $X_i(K_1, K_2) = 1$  if and only if  $|K_1 \cap K_2| = k - i$ , for  $i = 0, 1, \dots, k$ . In other words,  $X_i = U_{k,k}^{k-i}$ . In this section we introduce two new bases for the Bose-Mesner algebra of  $J(v, k)$  and obtain the associated intersection numbers.

**Remark 1.** Since the matrices of the association schemes are  $(0, 1)$  square matrices, we have to consider square matrices in this section. But the equations (i) and (iii) at the beginning of this section have the following analogues for non-square matrices

$$(U_{sk}^t)^\top = U_{ks}^t,$$

$$\sum_{\ell=0}^{\min(s,t,k)} U_{sk}^{t\ell} = J_{\binom{v}{t} \times \binom{v}{k}}.$$

The last equation is obtained by setting  $z = 0$  in (7) and (12).

The first new basis for the Bose-Mesner algebra of  $J(v, k)$  is  $\{A_{kk}^i : i = 0, \dots, k\}$ ; this follows from Equations (15) and (16). To introduce the second basis consider the matrix  $U_{sk}^{\geq \ell}$  defined as

$$U_{sk}^{\geq \ell}(S, K) := \begin{cases} 1, & \text{if } |S \cap K| \geq \ell; \\ 0, & \text{otherwise.} \end{cases}$$

Whence, we have  $U_{sk}^{\geq \ell} = \sum_{\ell'=\ell}^s U_{sk}^{\ell'}$  and  $U_{sk}^{\ell} = U_{sk}^{\geq \ell} - U_{sk}^{\geq \ell+1}$ . This shows that the matrices  $\{U_{kk}^{\geq \ell} : \ell = 0, \dots, k\}$  form a basis for the Bose-Mesner algebra of  $J(v, k)$ . The relation between the two new bases is also demonstrate the relation below.

**Proposition 12.** *If  $\ell > 0$ , then*

$$U_{sk}^{\geq \ell} = \sum_{i=\ell}^s (-1)^{i-\ell} \binom{i-1}{\ell-1} A_{sk}^i. \quad (32)$$

**Proof.** Let  $G_{sk}(z) = F_{sk}(z-1)$  and  $H_{sk}(z) = \frac{1}{z-1}(G_{sk}(z) - G_{sk}(1))$ . Then  $G_{sk}(z) = \sum_{\ell=0}^s U_{sk}^{\ell} z^{\ell}$  and  $H_{sk}(z) = \sum_{i=1}^s A_{sk}^i (z-1)^{i-1}$ . Therefore

$$\begin{aligned} U_{sk}^{\geq \ell} &= \sum_{\ell'=\ell}^s U_{sk}^{\ell'} \\ &= [z^{s+\ell}] \left( \left( \sum_{i=0}^s z^i \right) G_{sk}(z) \right) \\ &= [z^{s+\ell}] \left( \left( \sum_{i=0}^s z^i \right) (G_{sk}(z) - G_{sk}(1)) \right) \\ &= [z^{s+\ell}] \left( (z^{s+1} - 1) H_{sk}(z) \right) \\ &= [z^{s+\ell}] (z^{s+1} H_{sk}(z)) \\ &= \left[ \frac{d^{\ell-1}}{dz^{\ell-1}} H_{sk}(z) \right]_{z=0} \\ &= \sum_{i=\ell}^s (-1)^{i-\ell} \binom{i-1}{\ell-1} A_{sk}^i. \end{aligned}$$

□

Define the intersection numbers  $r_{ij}^{\ell}$  and  $p_{ij}^{\ell}$  as

$$A_{kk}^i A_{kk}^j = \sum_{\ell=0}^k r_{ij}^{\ell} A_{kk}^{\ell}, \quad \text{and} \quad U_{kk}^i U_{kk}^j = \sum_{\ell=0}^k p_{ij}^{\ell} U_{kk}^{\ell}.$$

From (19) and (20) we obtain

**Proposition 13.** *The values of parameters  $r_{ij}^\ell$  and  $p_{ij}^\ell$  are as follows:*

$$r_{ij}^\ell = \binom{v-i-j}{k-i-j+\ell} \binom{k-\ell}{i-\ell} \binom{k-\ell}{j-\ell} \text{ and } p_{ij}^\ell = \sum_{e=0}^{\ell} \binom{\ell}{e} \binom{k-\ell}{i-e} \binom{k-\ell}{j-e} \binom{v-2k+\ell}{k-i-j+e}.$$

## 8 Eigenvalues and rank of intersection matrices

The eigenvalues of the matrix  $U_{k,k}^{k-\ell}$ , for  $\ell = 0, 1, \dots, k$ , of the Johnson scheme  $J(v, k)$  can be expressed in terms of ‘‘Eberlein polynomials’’ (see [1, 3]) which are

$$\sum_{i=0}^{\ell} (-1)^{\ell-i} \binom{k-i}{\ell-i} \binom{k-j}{i} \binom{v-k+i-j}{i},$$

with multiplicity  $\binom{v}{j} - \binom{v}{j-1}$ , for  $j = 0, 1, \dots, k$ . In this section, we modify the Wilson’s proof of Lemma I of [13] (cf. [12, 14]) to obtain the eigenvalues of  $F_{kk}^t(z)$  and  $U_{kk}^{t\ell}$  as well as  $U_{kk}^{\geq \ell}$ . Moreover, we give a closed form for the eigenvalues and the rank of  $N_{kk}^{k-1}$ . The rank of  $U_{tk}^\ell$  is also investigated.

The following lemma which gives the eigenvalues and the corresponding eigenvectors of  $A_{kk}^i$  is proved in [12] and its proof is based on Proposition 13. The following decomposition of  $\mathbb{R}^{\binom{v}{k}}$  is used in the lemma: Fix  $k$  and let  $R_j$  denote the row-space of  $W_{jk}$  over the field  $\mathbb{R}$ . As mentioned before  $R_0 \subseteq R_1 \subseteq \dots \subseteq R_k = \mathbb{R}^{\binom{v}{k}}$ . Now let  $V_0 = R_0$ , and  $V_j := R_j \cap R_{j-1}^\perp$  for  $j = 1, \dots, k$ . Evidently  $\dim(R_j) = \binom{v}{j} - \binom{v}{j-1}$ .

**Lemma 14.** *With the above definitions, for any  $\mathbf{x}_j \in V_j$ , one has  $A_{kk}^i \mathbf{x}_j^\top = \lambda_j \mathbf{x}_j^\top$ , where*

$$\lambda_j = \begin{cases} \binom{v-i-j}{k-i} \binom{k-j}{i-j}, & \text{if } i \geq j; \\ 0, & \text{otherwise.} \end{cases}$$

*In other words, the vectors of  $R_i^\perp$  are eigenvectors of the value 0 for  $A_{kk}^i$  and the vectors in  $V_j$ ,  $0 \leq j \leq i$  are eigenvectors for the value  $\binom{k-j}{i-j} \binom{v-j-i}{k-i}$ .*

The following theorem determines the eigenvalues of  $F_{kk}^t(z)$ . Before this we need some further definitions: Fix  $k$  and let  $R_j(z)$  denote the row-space of  $W_{jk}$  over the field of rational functions  $\mathbb{R}(z)$ . and let  $V_0(z) = R_0(z)$ , and  $V_j(z) := R_j(z) \cap R_{j-1}(z)^\perp$  for  $j = 1, \dots, k$ . It is easy to prove that a basis of  $R_j$  (resp.  $V_j$ ) can also be considered as a basis of  $R_j(z)$  (resp.  $V_j(z)$ ).

**Theorem 15.** *Let  $0 \leq t \leq k \leq v/2$ . Consider  $F_{kk}^t(z)$  as a matrix with entries in the field of rational functions  $\mathbb{R}(z)$ . Then the eigenvalues of  $F_{kk}^t(z)$  are*

$$\mu_0(z)^{\binom{v}{0}}, \mu_1(z)^{\binom{v}{1}-\binom{v}{0}}, \dots, \mu_t(z)^{\binom{v}{t}-\binom{v}{t-1}}, 0^{\binom{v}{k}-\binom{v}{t}},$$

where the exponents indicate the multiplicity and

$$\mu_j(z) = \sum_{i=j}^t \binom{k-j}{i-j} \binom{v-j-i}{k-i} z^i, \quad (33)$$

for  $j = 0, 1, \dots, t$ . Furthermore, with the above notations, the vectors in  $V_j(z)$ ,  $0 \leq j \leq t$  are eigenvectors for the value  $\mu_j$ , for  $j = 0, \dots, t$ . The other eigenvectors (which correspond to the eigenvalue 0) belong to  $R_t(z)^\perp$ .

**Proof.** Considering  $F_{kk}^t(z) = \sum_{i=0}^t A_{kk}^i z^i$  the proof follows from Lemma 14.  $\square$

**Corollary 16.** Let  $0 \leq t \leq k \leq v/2$ . The eigenvalues of  $U_{kk}^{t\ell}$  are

$$\lambda_0^{\binom{v}{0}}, \lambda_1^{\binom{v}{1} - \binom{v}{0}}, \dots, \lambda_t^{\binom{v}{t} - \binom{v}{t-1}}, 0^{\binom{v}{k} - \binom{v}{t}},$$

where

$$\lambda_j = \sum_{i=\ell}^t (-1)^{\ell+i} \binom{i}{\ell} \binom{k-j}{i-j} \binom{v-j-i}{k-i}, \quad (34)$$

for  $j = 0, 1, \dots, t$ .

**Proof.** This corollary can be proved using (15) and Lemma 14. An alternative proof is obtained by using theorem 15: According to this theorem, the space of eigenvectors of a given eigenvalue of  $F_{kk}^t(z)$  has a basis independent of  $z$ . Thus the eigenvalues of  $\frac{D^\ell}{\ell!} F_{kk}^t(z)$  are obtained by the action  $\frac{D^\ell}{\ell!}$  on the eigenvalues of  $F_{kk}^t(z)$ . The result is then provided by setting  $z = -1$ .  $\square$

In view of (32), the following follows from Lemma 14.

**Corollary 17.** Let  $\ell > 0$  and  $k \leq v/2$ . The eigenvalues of  $U_{kk}^{\geq \ell}$  are

$$\sum_{i=\ell}^k (-1)^{\ell+i} \binom{i-1}{\ell-1} \binom{k-j}{i-j} \binom{v-j-i}{k-i},$$

with multiplicity  $\binom{v}{j} - \binom{v}{j-1}$ , for  $j = 0, 1, \dots, k$ .

By Corollary 16, the eigenvalues of  $U_{kk}^{t,0}$  are

$$\lambda_j = (-1)^k \binom{2k-v-1}{k-j} - \sum_{i=t+1}^k (-1)^i \binom{k-j}{i-j} \binom{v-j-i}{k-i}.$$

The case  $k - t = 1$  is discussed in the following.

**Corollary 18.** Let  $k \leq v/2$ . Then

(i) the eigenvalues of the matrix  $N_{kk}^{k-1}$  are

$$\lambda_0^{(v)} = \lambda_1^{(v) - \binom{v}{1}}, \dots, \lambda_{k-1}^{(v) - \binom{v}{k-2}}, 0^{(v) - \binom{v}{k-1}},$$

where

$$\lambda_j = 1 - \binom{2k - v - 1}{k - j}, \text{ for } j = 0, 1, \dots, k - 1;$$

(ii)  $\text{rank } N_{kk}^{k-1}(2k) = \frac{1}{2} \binom{2k}{k}$ ;

(iii)  $\text{rank } N_{kk}^{k-1}(v) = \binom{v}{k-1}$  provided that  $k < v/2$ .

**Proof.** (i) Consider  $N_{kk}^{k-1} = (-1)^{k-1} U_{kk}^{k-1,0}$  and use Corollary 16.

(ii) Setting  $v = 2k$  in part (i) yields  $\lambda_j = 1 + \binom{-1}{k-j} = 1 + (-1)^{k-j+1}$  for  $j = 0, 1, \dots, k-1$ . Hence  $\lambda_j \neq 0$  if and only if  $2 \mid j + k - 1$ . Therefore

$$\text{rank } N_{kk}^{k-1}(2k) = \sum_{\substack{0 \leq j \leq k-1 \\ 2 \mid j+k-1}} \left( \binom{2k}{j} - \binom{2k}{j-1} \right).$$

The result now follows from the identities  $\sum_j \binom{2k}{j} = 2^{2k-1}$  and  $2 \sum_j \binom{2k}{j-1} = 2^{2k} - \binom{2k}{k}$ , where  $j$  runs over the same set as in the above sum.

(iii) It is easily seen that in this case  $\lambda_j \neq 0$  for all  $j = 0, 1, \dots, k-1$ , thus  $\text{rank } N_{kk}^{k-1}(v) = \sum_{j=0}^{k-1} \left( \binom{v}{j} - \binom{v}{j-1} \right) = \binom{v}{k-1}$ .

□

**Example 3.** For the matrix  $N_{7,7}^6(14)$ ,  $\lambda_j = 1 - \binom{-1}{7-j} = 1 + (-1)^j$ , for  $j = 0, 1, \dots, 6$ . Thus the set of eigenvalues of  $N_{7,7}^6(14)$  is  $\{2^{1716}, 0^{1716}\}$ . So  $N_{7,7}^6(14)$  is a square matrix of order 3432 and rank 1716. For the matrix  $N_{6,6}^5(13)$ ,  $\lambda_j = 1 - \binom{-2}{6-j} = 1 + (-1)^{j+1}(7-j)$ , for  $j = 0, 1, \dots, 6$ . Thus the set of eigenvalues of  $N_{6,6}^5(13)$  is

$$\{(-6)^1, 7^{12}, (-4)^{65}, 5^{208}, (-2)^{429}, 3^{572}, 0^{429}\}.$$

So  $N_{6,6}^5(13)$  is a square matrix of order 1716 and rank 1287.

**Theorem 19.** Let  $0 \leq t \leq s \leq k \leq v/2$ . Consider  $F_{sk}^t(z)$  as a matrix with entries in the field of rational functions  $\mathbb{R}(z)$ . Then the eigenvalues of the matrix  $W_{sk}^\top F_{sk}^t(z)$  are

$$\alpha_0(z)^{(v)} = \alpha_1(z)^{(v) - \binom{v}{1}}, \dots, \alpha_t(z)^{(v) - \binom{v}{t-1}}, 0^{(v) - \binom{v}{t}},$$

where with notations of Theorem 15 we have  $\alpha_j(z) = L_{ks} \mu_j(z)$  and thus

$$\alpha_j(z) = (-1)^{k+s} \sum_{i=j}^t \binom{k-j}{i-j} \binom{v-j-i}{k-i} \binom{i-s-1}{k-s} z^i. \quad (35)$$

**Proof.** Again we remark that in theorem 15 the space of eigenvectors of a given eigenvalue of  $F_{kk}^t(z)$  has a basis independent of  $z$ . From this and the equation  $W_{sk}^\top F_{sk}^t = L_{ks} F_{kk}^t$  it is concluded that the eigenvalues of  $W_{sk}^\top F_{sk}^t$  are of the following form

$$(L_{ks}\mu_0(z))_{\binom{v}{0}}, (L_{ks}\mu_1(z))_{\binom{v}{1}-\binom{v}{0}}, \dots, (L_{ks}\mu_t(z))_{\binom{v}{t}-\binom{v}{t-1}}, 0_{\binom{v}{k}-\binom{v}{t}}$$

and this yields the result.  $\square$

**Corollary 20.** *Let  $0 \leq s \leq k \leq v/2$ . The eigenvalues of the matrix  $W_{sk}^\top U_{sk}^\ell$  are*

$$\tau_0^{\binom{v}{0}}, \tau_1^{\binom{v}{1}-\binom{v}{0}}, \dots, \tau_s^{\binom{v}{s}-\binom{v}{s-1}}, 0_{\binom{v}{k}-\binom{v}{s}},$$

where

$$\tau_j := (-1)^{k+s+\ell} \sum_{i=\min(j,\ell)}^k (-1)^i \binom{i}{\ell} \binom{k-j}{i-j} \binom{v-j-i}{k-i} \binom{i-s-1}{k-s}, \quad (36)$$

for  $j = 0, 1, \dots, s$ . Hence

$$\text{rank } U_{sk}^\ell = \sum_{\substack{0 \leq j \leq s \\ \tau_j \neq 0}} \left( \binom{v}{j} - \binom{v}{j-1} \right).$$

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